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A relation between the solutions of the half-space Dirichlet problems for Helmholtz's equation in $\mathbb{R}^n$ and Laplace's equation in $\mathbb{R}^{n+1}$

I. N. SNEDDON

Department of Mathematics, University of Glasgow, Glasgow (Scotland)

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SUMMARY
Multiple Fourier transforms are used to derive the solutions of the half-space Dirichlet problems for Helmholtz's equation in R^n and Laplace's equation in R^{n+1} and to exhibit the relation between the two solutions.

1. Introduction

In a recent paper Boudjelkha and Diaz [1] used Hadamard's method of descent to show how to derive the solution of the half-space Dirichlet problem for Helmholtz's equation in $\mathbb{R}^n$ from that of the corresponding problem for Laplace's equation in $\mathbb{R}^{n+1}$. The purpose of this brief note is to show that their formulae may be derived easily by the use of the theory of multiple Fourier transforms.

2. Formulation of the problems

We shall use the notation $\boldsymbol{x}=(x_1, x_2, \ldots, x_{n-1})$ for a vector in $\mathbb{R}^{n-1}$ and $(\boldsymbol{x}, z)=(x_1, \ldots, x_{n-1}, z)$ and $(\boldsymbol{x}, y, z)=(x_1, \ldots, x_{n-1}, y, z)$ for vectors in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ respectively. The Laplacian operators Δ_n and Δ_{n+1} are defined by the equations

$$\Delta_n = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_{n+1} = \Delta_n + \frac{\partial^2}{\partial y^2}$$

respectively.

We consider the relation between the solution $v(\boldsymbol{x}, z)$ of the half-space Dirichlet problem

$$\begin{aligned} &(\Delta_n - \lambda^2)v(\boldsymbol{x}, z) = 0, \quad z>0 \\ &v(\boldsymbol{x}, 0) = g(\boldsymbol{x}) \\ &v(\boldsymbol{x}, z) \to 0 \text{ as } |\boldsymbol{x}^2+z^2| \to \infty, \ z>0 \end{aligned} \tag{2.1}$$

for the Helmholtz equation in $\mathbb{R}^n$ and the solution $w(\boldsymbol{x}, y, z)$ of the half-space Dirichlet problem

$$\begin{aligned} &w(\boldsymbol{x}, y, z) = 0, \quad z>0 \\ &w(\boldsymbol{x}, y, 0) = f(\boldsymbol{x}, y) \\ &w(\boldsymbol{x}, y, z) \to 0 \text{ as } |\boldsymbol{x}^2+y^2+z^2| \to \infty, \quad z>0 \end{aligned} \tag{2.2}$$

for the Laplace equation in $\mathbb{R}^{n+1}$; the functions f and g are assumed to be prescribed.

We first of all solve these equations by the use of multiple Fourier transforms using the notation

$$\hat{\phi}(\boldsymbol{\xi}) \equiv \mathscr{F}_{(n-1)}[\phi(\boldsymbol{x}); \boldsymbol{\xi}] = (2\pi)^{-\frac{1}{2}(n-1)} \int_{\mathbb{R}^{n-1}} \phi(\boldsymbol{x}) \exp\{i(\boldsymbol{\xi}\cdot\boldsymbol{x})\}\,d\boldsymbol{x} \tag{2.3}$$

$$\hat{\phi}(\boldsymbol{\xi}, \eta) \equiv \mathscr{F}_{(n)}[\phi(\boldsymbol{x}, y); (\boldsymbol{\xi}, \eta)] = (2\pi)^{-\frac{1}{2}n} \int_{\mathbb{R}^n} \phi(\boldsymbol{x}, y) \exp\{i(\boldsymbol{\xi}\cdot\boldsymbol{x}+\eta y)\}\,d\boldsymbol{x}\,dy \tag{2.4}$$

where $\xi=(\xi_1, \ldots, \xi_{n-1})$ and $\xi \cdot x$ denotes the inner product $\xi_1 x_1 + \ldots + \xi_{n-1} x_{n-1}$. The inverses $\mathscr{F}^*_{(n-1)}, \mathscr{F}^*_{(n)}$ of the operators $\mathscr{F}_{(n-1)}, \mathscr{F}_{(n)}$ are given respectively by

$$\phi(x) \equiv \mathscr{F}^*_{(n-1)}[\bar{\phi}(\xi); x] = \mathscr{F}_{(n-1)}[\bar{\phi}(\xi); -x] \tag{2.5}$$

and

$$\phi(x, y) \equiv \mathscr{F}^*_{(n)}[\bar{\phi}(\xi, \eta); (x, y)] = \mathscr{F}_{(n)}[\bar{\phi}(\xi, \eta); -(x, y)]\,. \tag{2.6}$$

3. Solution of the Dirichlet problem for Laplace's equation

To find the solution of the Dirichlet problem (2.1) in the half-space $z>0$ we operate on both sides of each of the equations (2.1) by $\mathscr{F}_{(n-1)}$ and make use of the result

$$\mathscr{F}_{(n-1)}[\Delta_n v(x, z); \xi] = \left(\frac{\partial^2}{\partial z^2} - \xi^2\right) \hat{v}(\xi, z)$$

where $\xi^2 = \xi_1^2 + \ldots + \xi_{n-1}^2$ and

$$\hat{v}(\xi, z) = \mathscr{F}_{(n-1)}[v(x, z); \xi]$$

(see e.g. p. 78 of [2]) we find that they are equivalent to the equations

$$\left[\frac{\partial^2}{\partial z^2} - \xi^2 - \lambda^2\right] \hat{v}(\xi, z) = 0$$

$$\hat{v}(\xi, 0) = \hat{g}(\xi)$$

$$\hat{v}(\xi, z) \to 0 \text{ as } z \to \infty$$

where

$$\hat{g}(\xi) = \mathscr{F}_{(n-1)}[g(x); \xi]\,.$$

These have solution

$$\hat{v}(\xi, z) = \hat{g}(\xi) \exp\{-(\xi^2 + \lambda^2)^{\frac{1}{2}} z\} \tag{3.1}$$

where the positive square root is taken. Using the convolution theorem for multiple Fourier transforms, (p. 79 of [2]), we can write this result in the form

$$v(x, z) = (2\pi)^{-\frac{1}{2}(n-1)} \int_{\mathrm{R}^{n-1}} g(s) L(x-s, z) \mathrm{d}s \tag{3.2}$$

where $s \in \mathrm{R}^{n-1}$ and the kernel $L(x, z)$ is defined by the equation

$$L(x, z) = \mathscr{F}^*_{(n-1)}[\exp\{-(\xi^2 + \lambda^2)^{\frac{1}{2}} z\}; x]\,. \tag{3.3}$$

Using a well-known result in the theory of integral transforms (p. 82 of [2]) we see that we can replace this formula by

$$L(x, z) = |x|^{-\nu} \mathscr{H}_\nu[r^\nu \exp\{-(r^2 + \lambda^2)^{\frac{1}{2}} z\}; |x|] \tag{3.4}$$

where $\nu = \frac{1}{2}(n-3)$ and $\mathscr{H}_\nu$ is the operator of the Hankel transform defined by the equation

$$\mathscr{H}_\nu[f(r); \rho] = \int_0^\infty r f(r) J_\nu(\rho r) \mathrm{d}r\,. \tag{3.5}$$

Making use of formula (19) on p. 31, Vol. II of [3] we find that

$$L(x, z) = \sqrt{(2/\pi)}\, \lambda^{\frac{1}{2}n} z (|x|^2 + z^2)^{-n+\frac{3}{4}} K_{\frac{1}{2}n}\{\lambda \sqrt{(|x|^2 + z^2)}\} \tag{3.6}$$

4. Solution of the Dirichlet problem for the Helmholtz equation

Similarly, we can show that the Dirichlet problem (2.2) has the solution $w(x, y, z)$ where $\hat{w}(\xi, \eta, z) \equiv \mathscr{F}_{(n)}[w(x, y, z); (\xi, \eta)]$ is of the form

$$\hat{w}(\xi, \eta, z) = \hat{f}(\xi, \eta) \exp\{-(\xi^2+\eta^2)^{\frac{1}{2}} z\} \tag{4.1}$$

where

$$\hat{f}(\xi, \eta) = \mathscr{F}_{(n)}[f(\boldsymbol{x}, y); (\boldsymbol{\xi}, \eta)] . \tag{4.2}$$

In other words

$$w(\boldsymbol{x}, y, z) = (2\pi)^{-\frac{1}{2}n} \int_{\mathbf{R}^n} f(\boldsymbol{s}, t) K(\boldsymbol{x}-\boldsymbol{s}, y-t, z) \mathrm{d}\boldsymbol{s} \mathrm{d}t \tag{4.3}$$

where

$$K(\boldsymbol{x}, y, z) = \mathscr{F}^*_{(n)}[\exp\{-(\xi^2+\eta^2)^{\frac{1}{2}} z\}; (\boldsymbol{x}, y)] \tag{4.4}$$

or

$$K(\boldsymbol{x}, y, z) = \rho^{-\nu} \mathscr{H}_\nu [r^\nu \mathrm{e}^{-rz}; \rho] \tag{4.5}$$

where $\nu = \frac{1}{2}n - 1$ and $\rho = \sqrt{(|\boldsymbol{x}|^2 + y^2)}$. Making use of formula (8) on p. 182, Vol. I of [3] we find that

$$K(\boldsymbol{x}, y, z) = 2^{\frac{1}{2}n} \pi^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}n + \tfrac{1}{2}) z (\rho^2 + z^2)^{-\frac{1}{2}n-1} , \quad \rho^2 = |\boldsymbol{x}|^2 + y^2 . \tag{4.6}$$

5. The relation between the solutions

We now consider the relation between the solutions derived in §§3 and 4.

If we take

$$f(\boldsymbol{x}, y) = g(\boldsymbol{x}) \mathrm{e}^{-i\lambda y} \tag{5.1}$$

in §4, i.e. take

$$\hat{f}(\boldsymbol{\xi}, \eta) = \sqrt{(2\pi)}\, \delta(\eta - \lambda) \hat{g}(\boldsymbol{\xi})$$

in equation (4.1) we find that

$$\hat{w}(\boldsymbol{\xi}, \eta, z) = \sqrt{(2\pi)}\, \hat{g}(\boldsymbol{\xi}) \exp\{-(\xi^2+\eta^2)^{\frac{1}{2}} z\} \delta(\eta - \lambda) .$$

From equation (3.1) we deduce that

$$\hat{w}(\boldsymbol{\xi}, \eta, z) = \sqrt{(2\pi)}\, \hat{v}(\boldsymbol{\xi}, z) \delta(\eta - \lambda) .$$

Applying the operator $\mathscr{F}^*_{(n)}$ to both sides of this equation we obtain the result

$$w(\boldsymbol{x}, y, z) = \mathrm{e}^{-i\lambda y} v(\boldsymbol{x}, z) .$$

In other words if $w(\boldsymbol{x}, y, z)$ is the solution of the Dirichlet problem

$$\begin{aligned} &\Delta_{n+1} w(\boldsymbol{x}, y, z) = 0 , \quad z > 0 \\ &\quad w(\boldsymbol{x}, y, 0) = g(\boldsymbol{x}) \mathrm{e}^{-i\lambda y} \\ &\quad w(\boldsymbol{x}, y, z) \to 0 \text{ as } |\boldsymbol{x}^2 + y^2 + z^2| \to \infty , \quad z > 0 \end{aligned} \tag{5.2}$$

then the solution of the Dirichlet problem (2.1) is

$$v(\boldsymbol{x}, z) = w(\boldsymbol{x}, y, z) \mathrm{e}^{-i\lambda y} . \tag{5.3}$$

From this equation and equations (4.3), (5.1) we deduce that the solution of the Dirichlet problem (2.1) may be written in terms of the kernel K by means of the formula

$$v(\boldsymbol{x}, z) = (2\pi)^{-\frac{1}{2}n} \int_{\mathbf{R}^n} g(\boldsymbol{s}) \mathrm{e}^{i\lambda(y-t)} K(\boldsymbol{x}-\boldsymbol{s}, y-t, z) \mathrm{d}\boldsymbol{s} \mathrm{d}t$$

which by a trivial change of variable reduces to

$$v(\boldsymbol{x}, z) = (2\pi)^{-\frac{1}{2}n} \int_{\mathbf{R}^n} g(\boldsymbol{s}) \mathrm{e}^{i\lambda y} K(\boldsymbol{x}-\boldsymbol{s}, y, z) \mathrm{d}\boldsymbol{s} \mathrm{d}y . \tag{5.4}$$

Comparing this equation with equation (3.2) we deduce that the kernels K and L are related by the formula

$$L(\mathbf{x}, z) = \mathscr{F}[K(\mathbf{x}, y, z); y \to \lambda] . \tag{5.5}$$

That we recover the formula (3.6) by inserting the form (4.6) for K in equation (5.5) is verified by formula (7) on p. 11 of Vol. I of [3].

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